applied to two model problems. In both cases excellent agreement is obtained between numerical prediction and analytical solution. The method is also capable of locating the shock accurately within two mesh points. Extension of the methodology to two space dimensions on a streamline grid system has been completed, and the findings will be reported in the near future.

Acknowledgments

This work was performed at the Computational Fluid Dynamics Unit of the Imperial College, London, and was supported by a Commonwealth Scholarship. The author wishes to thank Professor Brian Spalding and Dr. George Carroll for their helpful discussions and enthusiasm.

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Inviscid, Unsteady, Transonic Axisymmetric Flow in Nozzles with Shock Waves

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Introduction

KNOWLEDGE of the behavior of shock waves in unsteady transonic nozzle flow is important when investigating internal machinery flows. These transient conditions may occur because of combustion instabilities or changes in power settings, among other reasons.[1]

Two techniques have been used to handle this type of problem: similarity transformations[2,3] and asymptotic expansions.[1,4] In particular, Messiter and Adamson[4] discussed the case of a steady two-dimensional transonic flow and presented analytical results for an axisymmetric nozzle flow using the asymptotic technique. This Note reviews this analytical development of Messiter and Adamson[4] for the axisymmetric transonic flow and extends it to include unsteady conditions suggested by those authors and Richey.[3]

Governing Equations

Consider an unsteady, inviscid, transonic, axisymmetric chocked flow of a perfect gas as shown in Fig. 1. The governing equations can be obtained by imposing small perturbations in a steady, uniform, irrotational flowfield. Using the same nomenclature and procedure as Richey and Adamson,[1] one can make the flow velocity components $\bar{U}$ and $\bar{V}$ dimensionless with respect to the sonic speed $\bar{a}^*$. The longitudinal and radial coordinates $\bar{x}$ and $\bar{r}$ are normalized by $\bar{L}$ (the radius of the nozzle throat) and the time $\bar{T}$ by $\bar{L}/\bar{a}^*$.

Outer Region

The dimensionless velocity potential function,[1] $\Phi(x,r,t) \equiv \Phi(\bar{x},\bar{r},\bar{t})/\overline{La^*}$, can be asymptotically expanded in terms of E (where E can be seen as the square root of the ratio between the throat radius and the radius of the nozzle curvature at the throat) and, together with the Bernouilli equation, substituted in the gasdynamic equation[1] for the velocity potential (as $U=\Phi_x$ and $V=\Phi_r$). From symmetry, $V=0$ at $r=0$.

The wall profile[3] can be taken as $r_w=[1+E^2f(x,t)]$, where $f(0,t)=0$. The boundary conditions,[4] assuming stationary walls [$f=f(x)$ only] can be written as

$$\phi_{2r}\,|_{r=1}=f' \tag{1a}$$

$$\phi_{3r}\,|_{r=1}=\phi_{1x}\phi_{2r}\,|_{r=1} \tag{1b}$$

where subscripts x and r mean $\partial/\partial x$ and $\partial/\partial r$, respectively, the numbered subscripts designate the expansion order, and $f'=\mathrm{d}f/\mathrm{d}x$.

Further, to first order, $\phi_1=\phi_1(x,t)$ only and from the second-order term, together with Eq. (1a), one can write the first-order governing equations as

$$\frac{2k}{\gamma+1}\phi_{1xt}+\phi_{1x}\phi_{1xx}=\frac{2f'}{\gamma+1} \tag{2}$$

where γ is the specific heat ratio.

Substituting Eqs. (1b) and (2) into the third-order term[4] and integrating with respect to x, one obtains an equation for $h(x,t)$,

$$\left(\frac{3-2\gamma}{6}\right)\phi_{1x}^3+\frac{k}{4}\left(\frac{3-\gamma}{\gamma+1}\right)\int_x(\phi_{1x}^2)_t\,\mathrm{d}x$$
$$=f''\frac{\phi_{1x}}{4}+\frac{2k}{\gamma+1}h_t+k\left(\frac{\gamma-1}{\gamma+1}\right)\phi_{1x}\phi_{1t}$$
$$+\phi_{1x}h_x+A_1(t) \tag{3}$$

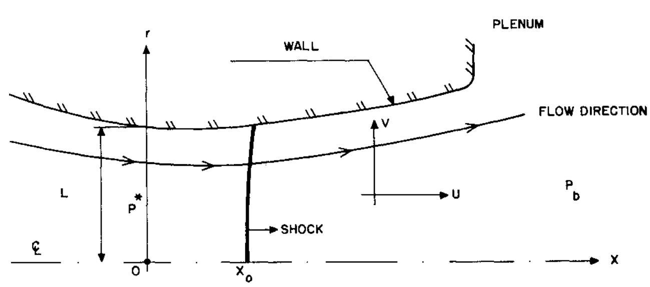

Fig. 1 Suggested nozzle flow geometry.

Received Sept. 9, 1987; revision received Aug. 31, 1988.

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where h is a constant of integration of ϕ_{3rr} and $A_1(t)$ depends on the initial given conditions.

From the second-order term of $\Phi(x,r,t)$, using Eq. (2) and taking the derivatives with respect to x and r one may write

$$u_2 = \phi_{2x} = f''\frac{r^2}{2} + h_x \tag{4a}$$

$$v_2 = \phi_{2r} = f'r \tag{4b}$$

Equations (2) and (3) yield ϕ_{1x} and h_x for given wall profile and initial conditions. This solution is valid everywhere except in the neighborhood of the throat and around the shock wave that might be present downsteam of the throat.[1,3,4] Then, it is necessary to introduce an inner solution to satisfy those conditions and match it with the outer solution decribed above. The flow is initially steady and small pressure disturbances are imposed downstream of the shock.[1] These disturbances tavel upstream into the nozzle toward the shock, exciting its motion.

Inner Region

Very close to the nozzle throat, the first- and the second-order solutions have the same order of magnitude. For the second- and higher-order solutions to satisfy the Rankine-Hugoniot conditions across the shock, it is necessary to assume an inner region enveloping the shock. This is done by stretching the longitudinal variable,[4] i.e., choosing $x^* = (x - X_{sh})E^{-1/2}$, where X_{sh} is the instantaneous shock location. The other variables, r and t, remain unchanged. The solution to be found has to match the outer solution, as $x^* \to \infty$. Further $X_{sh}(r,0) = x_0$, where x_0 is the initial shock location.

One can obtain the following expressions[1] for the velocity potential ϕ^* in the inner region, the instantaneous shock location $X_{sh}(r,t)$, and the shock speed along the x direction U_{sh} for this thin region of thickness of the order of $E^{1/2}$:

$$\phi^*(x^*,r,t) = E^{1/2}(x^* + E\phi_1^* + E^{3/2}\phi_{3/2}^* + E^2\phi_2^* + \ldots) \tag{5}$$

$$X_{sh}(r,t) = x_0 + Ex_{sh2}(t) + E^{3/2}x_{sh5/2}(r,t) + \ldots \tag{6}$$

$$U_{sh} = kE^2u_{sh2} + \ldots \tag{7}$$

where x_{sh2} in Eq. (6) is assumed to be a function of t only.

Substituting Eq. (5) and the Bernoulli equation into the gasdynamic equation for the velocity potential and using the inner stretched variables, the inner-region governing equations can be obtained for each order of magnitude.

These equations must satisfy the instantaneous Rankine-Hugoniot (R-H) jump conditions[1] and match with the outer solution as $|x^*| \to \infty$. Also, the flow must be tangent at the wall. Notice that at the neighborhood of the nozzle throat, it is not necessary, as shown by Adamson et al.,[5] that $E \to 0$ any more rapidly there.

Matching the Regions

If the flow is choked and steady to first order, then Eq. (2) can be integrated, yielding

$$\phi_{1x} = \pm\left[\frac{4}{\gamma+1}f(x)\right]^{1/2} \tag{8}$$

The outer solution can be expressed in terms of the inner variables, yielding equations for U and V that represent the mathematical matching term by term of the inner solution for large values of x^*. If this result is compared with the shock polar,[1,4] it is seen that the second- and higher-order solutions will not fulfill the continuity requirements. It would be convenient that the expression for u_{sh2} were a function of time only. This allows separation of the perturbation effect. Hence, one adds a second-order potential ζ^* to the inner solution.

Applying the overall solution (i.e., the sum of the inner and outer solutions) to the second-order inner-region governing equation and the boundary and matching (asymptotic) conditions, one obtains for ζ^* a Neumann-type boundary value problem,[1] which requires that

$$\int_0^1 \zeta_{x^*}^*(0,r,t)\,\mathrm{d}r = 0$$

yielding

$$\frac{4k}{\gamma+1}u_{sh2} = \frac{1}{2}f''(x_0) - \frac{4}{\gamma+1}f(x_0) + h_{xd} + h_{xu} \tag{9}$$

where subscripts u and d designate upstream or downstream of the shock, respectively.

The overall solution can be written as

$$U = 1 + E\phi_{1x} + E^2(\phi_{2x} + \zeta_{x^*}^*) + \ldots \tag{10a}$$

$$V = E^2(\phi_{2r} + E^{1/2}\zeta_r^*) + \ldots \tag{10b}$$

where $\zeta^* = 0$ upstream of the shock and ϕ_{1x}, ϕ_{2x}, and ϕ_{2r} are given by Eqs. (8) and (4).

Initial Value Problem

As $\phi_{1x} = \phi_{1x}(x)$ and choosing variables q and s such that $q = \text{const}$ along the characteristics, then from Eq. (3) it can be written

$$\left(\frac{\partial h}{\partial s}\right)_q = -A_1(t) - G(x) \tag{11}$$

where

$$\left(\frac{\partial x}{\partial s}\right)_q = \phi_{1x}, \quad \left(\frac{\partial t}{\partial s}\right)_q q = 2k/(\gamma+1)$$

$$G(x) = \left(\frac{\phi_{1x}}{2}\right)\left[\frac{2\gamma-3}{3}\phi_{1x}^2 + \frac{f''(x_0)}{2}\right]$$

Further, to go from $h(x,t)$ to $h(q,s)$ one writes

$$\left(\frac{\partial h}{\partial x}\right)_t = -\frac{1}{\bar{g}}\int_0^s \left(\bar{g}\frac{\partial F}{\partial x}\right)\mathrm{d}s + \frac{1}{\bar{g}}\left(\frac{\partial h_0}{\partial r}\right)_{t=0} \tag{12}$$

where

$$\bar{g}(q,s) \equiv \left(\frac{\partial x}{\partial q}\right)_s = \exp\left[\int_0^s \phi_{1xx}\mathrm{d}s\right]$$

and $h_0 = h(q,0)$ is the initial condition transmitted along the characteristics q with increasing s. One also chooses $k = (\gamma+1)/2$.

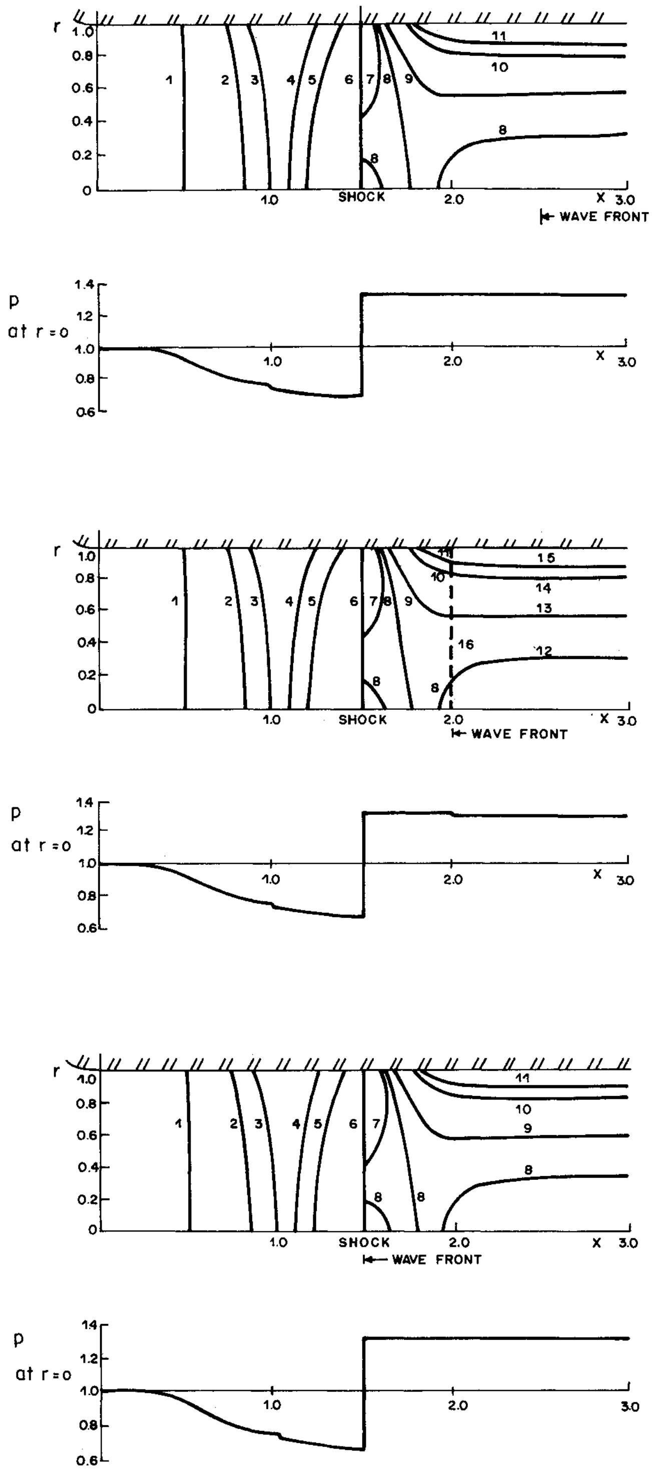
Assuming that the initial perturbation is given by

$$h_x(q,0) = (h_x)_{ss} + H(q-\hat{q})\cdot\epsilon\,\sin Nt \tag{13}$$

where $H(q-\hat{q})$ is the generalized step function, $\hat{q}$ the location of the disturbance such that $x = \hat{q}$ at $t = 0$, and the subscript ss stands for steady state. Notice that the values of q are related to x along $t = 0$ for $q > \hat{q}$ and to t along $x = \text{const}$ for $q \geq \hat{q}$.

The solution for $(h_x)_{ss}$ can be obtained from Eq. (11) with $h_t = 0$ and $A_1(t) = 2A(t) = \text{const}$, for convenience. Notice that this constant takes different values upstream and downstream of the shock. In this case,[4] $A_u = 0$ and $A_d = (1/3)\gamma(\phi_{1x}^3)_u$ as seen from Eq. (9), with $u_{sh2} = 0$, the steady-state situation being the initial condition.

For a given wall profile, the initial shock position x_0 is associated with the back pressure p_B specified at a station x_B far downstream.[1]



CURVE		MACH
1	.....	1.058
2	.....	1.158
3	.....	1.178
4	.....	1.205
5	.....	1.216
6	.....	1.000
7	.....	0.767
8	.....	0.764
9	.....	0.761
10	.....	0.756
11	.....	0.754

CASE a

CURVE		MACH
1	.....	1.058
2	.....	1.158
3	.....	1.178
4	.....	1.205
5	.....	1.216
6	.....	1.000
7	.....	0.767
8	.....	0.764
9	.....	0.761
10	.....	0.756
11	.....	0.754
12	.....	0.781
13	.....	0.778
14	.....	0.773
15	.....	0.771
16	.....	WAVE FRONT

CASE b

CURVE		MACH
1	.....	1.058
2	.....	1.158
3	.....	1.178
4	.....	1.205
5	.....	1.216
6	.....	1.000
7	.....	0.798
8	.....	0.795
9	.....	0.792
10	.....	0.787
11	.....	0.785

CASE c

Fig. 2 Constant Mach lines and pressure distribution at $r = 0$ for flow through a nozzle given by $r_w = \pm [1 + E^2 f(x)]$, where $f(x) = 1.5x^2$ for $x \leq 1$, $f(x) = -1.5\,(x-2)^2 + 3$ for $1 < x < 2$, and $f(x) = 3$ for $x \geq 2$ with $E = 0.1$ at: a) $t = 0.0$ b) $t = 0.22$, and c) $t = t_{sh} = 0.45$.

Numerical Results

Selecting a smooth wall profile, h_x can be obtained by integrating Eq. (12) numerically along characteristics $q = \text{const}$. Once h_x is known, ϕ_{2x} and ϕ_{2r} can be calculated, yielding U, V, and u_{sh2}, which is given by $u_{sh2} = 0.5\epsilon \sin Nt$, $t > t_{sh}$ (t_{sh} being the time taken for the disturbance to hit the shock). Choosing $\epsilon = 4.0$ and $N = 2.0$, the results shown in Fig. 2 for the Mach lines at three different times are obtained along the nozzle with the respective p vs x curves. Notice that the perturbation was taken to begin at station $\hat{q} = x = 2.5$. The shock was located initially at $x_0 = 1.5$.

Figure 2a shows the constant Mach lines at $t = 0$. Because, at this instant, the imposed perturbation signal is zero, they correspond to the steady-state conditions. Although the shock was taken to be plane, it has a small curvature, to second order, which causes the flow behind it to decelerate first and then to accelerate locally near the walls. These results fairly match those obtained by Lin and Chen.[6] At $t = 0.22$, as shown in Fig. 2b, the wave train has moved upstream to station $x = 2.01$. Notice that, due to the decreasing pressure, the Mach lines downstream from this section display a constant level jump. The perturbation front hits the shock wave at $x = 1.50$ in a time $t = t_{sh} = 0.45$, as shown in Fig. 2c. From that point, the flow properties everywhere downstream of the shock are functions of time, and the shock will start its motion under the influence of the perturbation signal. Figure 3 shows the shock velocity and position as functions of time.

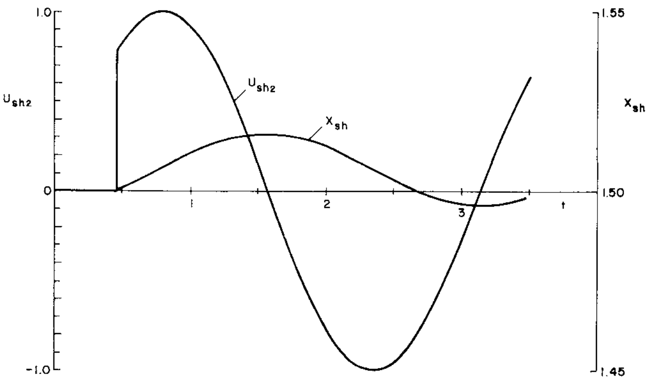


Fig. 3 Shock velocity U_{sh} and position X_{sh} vs time.

Conclusions

This analysis gives a qualitative insight on the nature of unsteady transonic nozzle flow without resorting to the more complicated numerical solutions of transport equations. It was done employing the same technique Richey and Adamson[1] applied to channel flows, and it also can be used to study two-dimension asymmetric flows. It should be noted that this approach is for flows across nozzles displaying a smooth change in cross section. Further, the applied perturbation should not possess a very high frequency since the adopted solution deals with slow time variation regimes. This leads to the choice of a characteristic time of the order of 10^{-2}–10^{-3} s.

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Finite-Element Method Applied to Transonic Flow Over a Bulbous Payload Shroud

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Introduction

ANALYSIS of transonic flow over a bulbous payload shroud of a launch vehicle requires accurate modeling of the shock boundaries as well as the details of the subsonic and supersonic regions and shock. A numerical analysis is also needed to complement wind-tunnel tests, when tunnel-wall interference may induce large deviations from free-air conditions even for very small blockage ratios. The efficiency of the numerical approach in analyzing the bulbous shape geometrical configuration is directly related to the applicability of irregular computational grids.

Numerical solution of steady, inviscid, and axisymmetric transonic flow over blunt and boat-tail configurations has been obtained using the finite-difference method by several researchers[1–4] during the last few years. During these investigations, the importance of computational grid, effect of artificial viscosity, and relaxation factor in the computation of transonic flow was observed. Application of the finite-element method for the analysis of two-dimensional inviscid transonic flow problem has been considered by several investigators.[5–7] One of the main advantages of the finite-element approach has been practicability of setting an irregular computational grid to fit a particular flow problem. This property therefore proves to be important for the solution of transonic flow over a bulbous payload shroud of a typical launch vehicle. The finite-element method has been applied by Chima and Gerhart[8] to the solution of subsonic flow over a boat-tail body.

The present work uses a mesh of triangular finite elements. This is chosen because it enables a large number of small elements to be packed into the regions of greatest interest such as in the vicinity of blunt body and boat-tail regions. The importance of designing a computational grid suitable for both subsonic regions and supersonic pockets to obtain accurate and efficient solution has been investigated in the present analysis. A shock-capturing technique is employed in conjunction with the pseudotime integration scheme.[9] The choice of artificial viscosity, relaxation factors, and design of a computational grid is investigated. It will be demonstrated that the finite-element method is a practical and efficient tool in treating complex flow problems. Comparisons between analysis and experiment have been made for two different types of axisymmetric geometry of bulbous payload shroud.

Full Potential Equations

The full potential equation for steady, inviscid, and irrotational flow in a conservation form for an axisymmetric body can be written as

$$(\rho r \phi_{,r})_{,r} + (\rho r \phi_{,z})_{,z} = 0 \qquad \text{on } v \tag{1}$$

where ϕ is the velocity potential, v the solution domain, and ρ the mass density of the fluid. Since no entropy can be created in irrotational flow, the density can be computed using the isentropic relation. Equation (1) is second-order quasilinear of elliptic type in subsonic, parabolic type in sonic, and hyperbolic type in supersonic points of flow.

The boundary condition at the solid surface is that the velocity normal to the surface equal to zero. At the downstream boundary of the computational field, the potential function ϕ is specified. The boundary conditions on the far field on an isolated bulbous payload shroud or body are only approximate. The correct boundary condition is that the velocities approach to freestream values far from the body. This boundary condition cannot be satisfied directly for a potential flow solution in a finite computational domain. The present far-field boundary conditions are satisfactory provided the boundaries are located far enough from the body. Locating the far-field boundaries 10–12 maximum body diameters away from the body is usually sufficient to insure minor effect of the boundary condition on the computed flowfield near the body. If the far-field boundary is so placed, the effect of the boundary conditions are small and are of the order of or smaller than other approximations such as treating a bulbous

Received Nov. 19, 1987; revision received July 20, 1988.

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